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Random matrix theory and the sixth Painlevé equation

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Abstract
A feature of certain ensembles of random matrices is that the corresponding measure is invariant under conjugation by unitary matrices. Study of such ensembles realized by matrices with Gaussian entries leads to statistical quantities related to the eigenspectrum, such as the distribution of the largest eigenvalue, which can be expressed as multidimensional integrals or equivalently as determinants. These distributions are well known to be τ-functions for Painlevé systems, allowing for the former to be characterized as the solution of certain nonlinear equations. We consider the random matrix ensembles for which the nonlinear equation is the σ form of P_{VI}. Known results are reviewed, as is their implication by way of series expansions for the distributions. New results are given for the boundary conditions in the neighbourhood of the fixed singularities at $t = 0, 1, \infty$ of σP_{VI} displayed by a generalization of the generating function for the distributions. The structure of these expansions is related to Jimbo's general expansions for the τ-function of σP_{VI} in the neighbourhood of its fixed singularities, and this theory is itself put in its context of the linear isomonodromy problem relating to P_{VI}.

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To commemorate the centenary of the publication of the Painlevé VI equation in the Comptes Rendus de l'Academie des Sciences de Paris by Richard Fuchs in 1905.

1. Introduction

1.1. The σ-form of Painlevé VI

Given a large sequence of an eigenvalue spectrum, it is a simple matter to rescale so that the mean spacing between consecutive eigenvalues is unity, then to empirically determine the distribution function for the spacing. When these eigenspectra are the highly excited states of heavy nuclei, it is a celebrated result that the distribution function is well approximated by the functional form

$$p_1^{(\mathrm{W})}(s) := \frac{\pi s}{2}\,\mathrm{e}^{-\pi s^2/4}, \tag{1.1}$$

known as the Wigner surmise. It is in the exact computation of eigenvalues distributions for certain classical random matrix ensembles that Painlevé transcendents make their appearance. These transcendents may relate to any of $\mathrm{P_{II}}$ to $\mathrm{P_{VI}}$, depending on the random matrix ensemble under consideration, or its scaled limits. As part of the centenary of the discovery of Painlevé VI by Fuchs [11], in this paper we will restrict attention to the random matrix ensembles relating to $\mathrm{P_{VI}}$.

Painlevé VI conventionally refers to the four-parameter second-order nonlinear differential equation

$$y'' = \frac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-t}\right)(y')^2 - \left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{y-t}\right)y' + \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha+\frac{\beta t}{y^2}+\frac{\gamma(t-1)}{(y-1)^2}+\frac{\delta t(t-1)}{(y-t)^2}\right), \tag{1.2}$$

as obtained by Fuchs [11]. However, in random matrix theory, this equation is never encountered directly. Rather what is encountered is the so-called Jimbo–Miwa–Okamoto σ-form of Painlevé VI,

$$\sigma'_{\mathrm{VI}}(t(t-1)\sigma''_{\mathrm{VI}})^2 + (\sigma'_{\mathrm{VI}}[2\sigma_{\mathrm{VI}} - (2t-1)\sigma'_{\mathrm{VI}}] + v_1v_2v_3v_4)^2 = \prod_{k=1}^{4}\left(\sigma'_{\mathrm{VI}} + v_k^2\right). \tag{1.3}$$

This is written in a form which displays a D_4 root system symmetry in the parameters $v_1,\ldots,v_4$. When expanded out there is a common factor of σ'_{VI}, which when cancelled out shows (1.3) to be a second-order second-degree nonlinear differential equation. Equations (1.2) and (1.3) are related by the Hamiltonian formulation of $\mathrm{P_{VI}}$, due originally to Malmquist in 1922 [19].

In the Hamiltonian approach to the Painlevé equations in general, one presents a Hamiltonian $H(q,p,t;\{v_k\})$, where $\{v_k\}$ are parameters, such that after eliminating p in the Hamilton equations,

$$q' = \frac{\partial H}{\partial p}, \qquad p' = -\frac{\partial H}{\partial q}, \tag{1.4}$$

where the dash denotes derivatives with respect to t, the equation in q is the appropriate Painlevé equation. The Hamiltonians can be systematically derived from the isomonodromy deformation theory associated with the Painlevé equations [16, 22] (aspects of the isomonodromy deformation theory associated with $\mathrm{P_{VI}}$ are covered in section 3 below). From such considerations, the Hamiltonian relating to $\mathrm{P_{VI}}$ was given by Okamoto [23] as

$$t(t-1)H_{\mathrm{VI}} = q(q-1)(q-t)p^2 - [(v_3+v_4)(q-1)(q-t) + (v_3-v_4)q(q-t) - (v_1+v_2)q(q-1)]p + (v_3-v_1)(v_3-v_2)(q-t) \tag{1.5}$$

where the parameters $v_1,\ldots,v_4$ are related to $\alpha,\beta,\gamma,\delta$ in (1.2) according to

$$\alpha = \tfrac{1}{2}(v_1-v_2)^2, \quad \beta = -\tfrac{1}{2}(v_3+v_4)^2, \quad \gamma = \tfrac{1}{2}(v_3-v_4)^2, \quad \delta = \tfrac{1}{2}(1-(1-v_1-v_2)^2), \tag{1.6}$$

and q satisfies (1.2). Note that H_{VI} is quadratic in p and thus according to the first of the Hamilton equations (1.4) p can be written as a rational function of q and q', and thus in fact

$H_{\rm VI}$ is a rational function in q and q'. According to the following result, this particular rational function, augmented by the addition of a linear function in t, satisfies (1.3) [23].

Proposition 1.1. *Define the auxiliary Hamiltonian*

$$h_{\rm VI}(t) = t(t-1)H_{\rm VI} + e_2[-v_1, -v_2, v_3]t - \tfrac{1}{2}e_2[-v_1, -v_2, v_3, v_4], \tag{1.7}$$

where

$$e_p[a_1, \ldots, a_s] := \sum_{1 \leqslant j_1 < \cdots < j_p \leqslant s} a_{j_1} a_{j_2} \cdots a_{j_p}. \tag{1.8}$$

This auxiliary Hamiltonian satisfies the σ-form of Painlevé VI (1.3).

1.2. Historical overview

There are certain ensembles of $N \times N$ random matrices with complex Gaussian entries and invariance under conjugation by unitary matrices, which have their joint eigenvalue probability distribution function (pdf) of the form

$$\frac{1}{C} \prod_{l=1}^{N} w_2(x_l) \prod_{1 \leqslant j < k \leqslant N} (x_k - x_j)^2, \tag{1.9}$$

where the weight function $w_2(x)$ is of one of the classical forms

$$w_2(x) = \begin{cases} e^{-x^2}, & \text{Gaussian} \\ x^a e^{-x} \quad (x > 0), & \text{Laguerre} \\ x^a (1-x)^b \quad (0 < x < 1), & \text{Jacobi.} \end{cases} \tag{1.10}$$

For example, let X be an $n \times N$ ($n \geqslant N$) rectangular matrix of complex Gaussians $\mathrm{N}[0, 1/\sqrt{(2)}] + \mathrm{i}\mathrm{N}[0, 1/\sqrt{(2)}]$. Then the matrix $X^\dagger X$ has eigenvalue pdf (1.9) with Laguerre weight $x^{n-N} e^{-x}$ $(x > 0)$. The sixth Painlevé equation relates to (1.9) with Jacobi weight, in particular to the probability that there are exactly n eigenvalues in the interval $(t, 1)$ of that ensemble. This probability is in turn equal to the coefficient of $(1-\xi)^n$ in the expansion of

$$E_N^J(t; a, b; \xi) := \frac{1}{C} \left(\int_0^1 - \xi \int_t^1 \right) \mathrm{d}x_1 \cdots \left(\int_0^1 - \xi \int_t^1 \right) \mathrm{d}x_N \\ \times \prod_{l=1}^{N} x_l^a (1-x_l)^b \prod_{1 \leqslant j < k \leqslant N} (x_k - x_j)^2, \tag{1.11}$$

where C denotes the normalization. In the case $N = 1$ equation (1.11) is an integral form of a particular ${}_2F_1$ hypergeometric function, which was related to the Painlevé VI equation by Okamoto [23].

Let $\{p_j(x)\}_{j=0,1,\ldots}$ denote the set of monic polynomials of degree j orthogonal with respect to the Jacobi weight $x^a(1-x)^b (0 < x < 1)$, which are given in terms of the Jacobi polynomials $P_j^{(a,b)}$ by

$$p_j(x) = (-1)^j j! \frac{\Gamma(a+b+j+1)}{\Gamma(a+b+2j+1)} P_j^{(a,b)}(1-2x). \tag{1.12}$$

Let $(p_j, p_j)_2 := \int_0^1 (p_j(x))^2 x^a (1-x)^b \, \mathrm{d}x$, and define

$$\tilde{K}_N^J(x, y) = \frac{(w_2(x) w_2(y))^{1/2}}{(p_{N-1}, p_{N-1})_2} \frac{p_N(x) p_{N-1}(y) - p_{N-1}(x) p_N(y)}{x - y}. \tag{1.13}$$

It is well known, and easy to derive (see, e.g., [4]), that with $\tilde{K}^J_{N,(t,1)}$ denoting the integral operator on $(t, 1)$ with kernel (1.13),

$$E^J_N(t; a, b; \xi) = \det\big(1 - \xi \tilde{K}^J_{N,(t,1)}\big). \tag{1.14}$$

It was in this form that

$$t(t-1)\frac{\mathrm{d}}{\mathrm{d}t}\log E^J_N \tag{1.15}$$

was first related to the solution of a nonlinear equation by Tracy and Widom [24]. The equation found was of third order. Subsequently Haine and Semengue [13] studied (1.11) itself in the case $\xi = 1$ and found a different third-order equation for (1.15). Upon subtracting the two equations they obtained a second-order second-degree nonlinear equation which they identified as an example of the σ-form of Painlevé VI (1.3). The study of Tracy and Widom proceeded via functional properties of quantities associated with the Fredholm determinant (1.14), while Haine and Semengue used the theory of the KP hierarchy and Virasoro constraints satisfied by certain matrix integrals as introduced by Adler and van Moerbeke [1]. A third approach to the problem was initiated by Borodin and Deift [3]. They combined Riemann–Hilbert theory with the method of isomonodromic deformation of certain linear differential equations to obtain a characterization of (1.15) which allows for immediate identification with the parameters in (1.3) (this is not the case with [13]). Explicitly it was shown that

$$\sigma(t) = -t(t-1)\frac{\mathrm{d}}{\mathrm{d}t}\log E^J_N(1-t; a, b; \xi) + v_1v_2t + \frac{1}{2}(-v_1v_2 + v_3v_4) \tag{1.16}$$

satisfies (1.3) with

$$v_1 = v_2 = N + \frac{a+b}{2}, \qquad v_3 = \frac{a+b}{2}, \qquad v_4 = \frac{a-b}{2}. \tag{1.17}$$

Furthermore, it is required that as $t \to 0$

$$\frac{\mathrm{d}}{\mathrm{d}t}\log E^J_N(1-t; a, b; \xi) \sim -\xi \tilde{K}^J_N(1-t, 1-t) \tag{1.18}$$

$$= -\xi C_N(a, b)(1-t)^b, \tag{1.19}$$

where

$$C_N(a, b) = \frac{\Gamma(a+b+N+1)\Gamma(b+N+1)}{\Gamma(N)\Gamma(a+N)\Gamma(b+1)\Gamma(b+2)}, \tag{1.20}$$

thus providing the boundary condition to be satisfied by (1.16).

According to (1.16),

$$E^J_N(1-t; a, b; \xi) = \exp\int_{1-t}^{1}\frac{\mathrm{d}s}{s(1-s)}\left(\sigma(s) - v_1v_2s - \frac{1}{2}(-v_1v_2 + v_3v_4)\right). \tag{1.21}$$

From the characterization of $\sigma(t)$ as a $\sigma\mathrm{P_{VI}}$ transcendent with a specific boundary condition, the power series solution of (1.21) about $t = 1$ can readily be computed [4],

$$\begin{aligned} E^J_N(1-t; a, b; \xi) = 1 - \xi\frac{C_N(a,b)}{b+1}(1-t)^{b+1} \\ \times\left\{1 - \frac{(b+1)(2N^2 + 2(a+b)N - 2 - 2b + ab)}{(b+2)^2}(1-t) + \mathrm{O}((1-t)^2)\right\} \\ + \xi^2\frac{C_N^2(a,b)(N-1)(N+b+1)(N+a-1)(N+a+b+1)}{(b+2)^2(b^2+4b+3)^2}(1-t)^{2b+4} \\ \times\{1 + \mathrm{O}(1-t)\}. \end{aligned} \tag{1.22}$$

Furthermore, one can anticipate from (1.14) that the leading term in $1-t$ accompanying the power ξ^k will be proportional to $(1-t)^{kb+k^2}$, as is consistent with (1.22).

It is well known (see, e.g., [4]) that after the change of variables $x_j = \cos^2\theta_j/2$, $0 \leqslant \theta_j < \pi$ and with $a, b = \pm 1/2$ the eigenvalue pdf for the Jacobi ensemble as specified by (1.9) and (1.10) becomes identical to the eigenvalue pdf for matrices from the classical groups $O^{\pm}(N)$, $Sp(2N)$ chosen with Haar (uniform) measure. As a consequence (in an obvious notation),

$$\begin{aligned} E^{O^-(2N+1)}((0,\phi);\xi) &= E_N^J(\cos^2\phi/2;\xi)\big|_{\substack{a=1/2\\ b=-1/2}} \\ E^{O^+(2N+1)}((0,\phi);\xi) &= E_N^J(\cos^2\phi/2;\xi)\big|_{\substack{a=-1/2\\ b=1/2}}. \end{aligned} \tag{1.23}$$

Analogous to expansion (1.22), the $\sigma\mathrm{P}_{\mathrm{VI}}$ evaluation (1.21) can then be used to deduce the expansions [4]

$$\begin{aligned} E^{O^-(2N+1)}((0,x);\xi) = 1 &- \tilde{c}x + \frac{4N^2-1}{36}\tilde{c}x^3 - \frac{48N^4-40N^2+7}{3600}\tilde{c}x^5 \\ &+ \frac{4N^4-5N^2+1}{2025}\tilde{c}^2x^6 + \frac{192N^6-336N^4+196N^2-31}{211\,680}\tilde{c}x^7 \\ &- \frac{48N^6-112N^4+77N^2-13}{198\,450}\tilde{c}^2x^8 + \mathrm{O}(x^9), \end{aligned} \tag{1.24}$$

$$\begin{aligned} E^{O^+(2N+1)}((0,x);\xi) = 1 &- \frac{4N^2-1}{36}\tilde{c}x^3 + \frac{(4N^2-1)(12N^2-7)}{3600}\tilde{c}x^5 \\ &- \frac{(4N^2-1)(48N^4-72N^2+31)}{211\,680}\tilde{c}x^7 + \mathrm{O}(x^9), \end{aligned} \tag{1.25}$$

where $\tilde{c} = 2N\xi/\pi$.

The method of [24] was adopted in [25] to relate

$$\begin{aligned} E_N^{Cy}(s;\eta;\xi) := \frac{1}{C}\left(\int_{-\infty}^{\infty} - \xi\int_s^{\infty}\right)\mathrm{d}x_1\cdots\left(\int_{-\infty}^{\infty} - \xi\int_s^{\infty}\right)\mathrm{d}x_N \\ \times \prod_{l=1}^{N}\frac{1}{(1+x_l^2)^{\eta}}\prod_{1\leqslant j<k\leqslant N}(x_k-x_j)^2 \end{aligned} \tag{1.26}$$

to $\sigma\mathrm{P}_{\mathrm{VI}}$. Explicitly it was shown that

$$\sigma(s) = (1+s^2)\frac{\mathrm{d}}{\mathrm{d}s}\log E_N^{Cy}(s;a+N;\xi) \tag{1.27}$$

satisfies the equation

$$\begin{aligned} (1+s^2)^2(\sigma'')^2 + 4(1+s^2)(\sigma')^3 - 8s\sigma(\sigma')^2 + 4\sigma^2(\sigma'-a^2) \\ + 8a^2s\sigma\sigma' + 4[N(N+2a)-a^2s^2](\sigma')^2 = 0. \end{aligned} \tag{1.28}$$

As noted in [7], the relationship between (1.28) and (1.3) can be seen by changing variables

$$t \mapsto \frac{\mathrm{i}s+1}{2}, \qquad \sigma_{\mathrm{VI}}(t) \mapsto \frac{\mathrm{i}}{2}h(s), \tag{1.29}$$

in the latter so that it reads

$$h'((1+s^2)h'')^2 + 4(h'(h-sh') - \mathrm{i}v_1v_2v_3v_4)^2 + 4\prod_{k=1}^{4}\left(h'+v_k^2\right) = 0. \tag{1.30}$$

With

$$h = \sigma - a^2 s, \quad v_1 = -a, \quad v_2 = 0, \quad v_3 = N + a, \quad v_4 = a, \tag{1.31}$$

(1.30) reduces to (1.28).

The interest in (1.26) in random matrix theory comes about by making a stereographic projection from the real line to the unit circle by the change of variables $x = \tan\theta/2$. With $z = e^{i\theta}$ this shows

$$\prod_{l=1}^{N} \frac{1}{\left(1 + x_l^2\right)^{N+\eta}} \prod_{1 \leqslant j < k \leqslant N} (x_k - x_j)^2 \, dx_1 \cdots dx_N \\ = 2^{-N(N+2\eta)} \prod_{l=1}^{N} |1 + z_l|^{2\eta} \prod_{1 \leqslant j < k \leqslant N} |z_k - z_j|^2 \, d\theta_1 \cdots d\theta_N. \tag{1.32}$$

In the case $\eta = 0$ the measure on the right-hand side of (1.32) corresponds to the eigenvalue pdf for $N \times N$ random unitary matrices chosen with Haar (uniform) measure. For general $\eta \in \mathbb{N}$ it, corresponds to this same ensemble conditional so that there is an eigenvalue of degeneracy η at $\theta = \pi$.

With $E_N^{U(N)}((\phi_1, \phi_2); \xi)$ denoting the generating function for the probability that the interval (ϕ_1, ϕ_2) contains exactly n eigenvalues, it follows that

$$E_N^{U(N)}((0, 2x); \xi) = \exp\left(-\int_0^x h(\cot\phi) \, d\phi\right), \tag{1.33}$$

where $h(s)$ satisfies (1.30) with $v_1 = v_2 = v_3 = 0, v_4 = N$. Since for $x \to 0$, $E_N^{U(N)}((0, 2x); \xi) \sim 1 - \xi x N/\pi$, we seek the solution of (1.30) subject to the boundary condition $h(s) \sim c, c := \xi N/\pi$. This allows the power series expansion

$$E_N^{U(N)}((0, 2x); \xi) = 1 - cx + \frac{N^2 - 1}{36} c^2 x^4 - \frac{(N^2 - 1)(2N^2 - 3)}{1350} c^2 x^6 \\ + \frac{(N^2 - 1)(N^2 - 2)(3N^2 - 5)}{52\,920} c^2 x^8 - \frac{(N^2 - 4)(N^2 - 1)^2}{291\,600} c^3 x^9 + O(x^{10}) \tag{1.34}$$

to be computed [4]. This expansion was first computed in [24] using the characterization of $E_N^{U(N)}$ in terms of a third-order nonlinear differential equation.

We have given reference to three distinct approaches which relate (1.11) to nonlinear differential equations with the Painlevé property. There is a fourth approach, which is due to the present authors [5–7], and involves applying Okamoto's theory of the Hamiltonian systems approach to P_{VI} [23]. This approach has the advantage of allowing generalizations of the generating functions (1.11), (1.26) and $E_N^{U(N)}((\phi_1, \phi_2); \xi)$ to be related to σP_{VI}. Consider for definiteness the latter. In [7] the more general quantity

$$A_N(t; \omega_1, \omega_2, \mu; \xi^*) := \frac{1}{N!} \left(\int_{-\pi}^{\pi} - \xi^* \int_{\pi-\phi}^{\pi}\right) \frac{d\theta_1}{2\pi} \cdots \left(\int_{-\pi}^{\pi} - \xi^* \int_{\pi-\phi}^{\pi}\right) \frac{d\theta_N}{2\pi} \\ \times \prod_{l=1}^{N} z_l^{-i\omega_2} |1 + z_l|^{2\omega_1} |1 + t z_l|^{2\mu} \prod_{1 \leqslant j < k \leqslant N} |z_j - z_k|^2, \tag{1.35}$$

where $t = e^{i\phi}, \phi \in [0, 2\pi), \xi^* \in \mathbb{C}$ and the parameters $\omega_1, \omega_2, \mu \in \mathbb{C}, \omega = \omega_1 + i\omega_2$, are restricted with $\Re(2\omega_1), \Re(2\mu) > -1, N \in \mathbb{Z}_{\geqslant 0}$. The independent variable t, whilst originally defined on the unit circle $|t| = 1$ with a real angle ϕ, can be considered as a complex variable which is analytically continued into the cut complex t-plane. The case $\omega_2 = \mu = 0, \omega_1 = \eta$

of (1.35) gives the generating function for the probability of k eigenvalues in $(0, \phi)$ for the ensemble specified by the right-hand side of (1.32). Define

$$M_N(a,b) := \int_{-1/2}^{1/2} \mathrm{d}x_1 \ldots \int_{-1/2}^{1/2} \mathrm{d}x_N \prod_{l=1}^{N} z_l^{(a-b)/2} |1+z_l|^{a+b} \prod_{1\leqslant j<k\leqslant N} |z_j - z_k|^2 \tag{1.36}$$

$$= \prod_{j=0}^{N-1} \frac{\Gamma(a+b+j+1)\Gamma(j+2)}{\Gamma(a+j+1)\Gamma(b+j+1)}, \qquad z_l = \mathrm{e}^{2\pi \mathrm{i} x_l}, \quad l = 1, \ldots, N, \tag{1.37}$$

set $\xi^* = 1 - (1-\xi)\,\mathrm{e}^{-\pi \mathrm{i}\mu}$ and denote by $\tilde{h}(s)$ a solution of (1.30) with

$$v_1 = -\mu - \omega_1, \quad v_2 = \mathrm{i}\omega_2, \quad v_3 = N + \mu + \omega_1, \quad v_4 = -\mu + \omega_1. \tag{1.38}$$

It was shown in [7] that

$$A_N(t; \omega_1, \omega_2, \mu; \xi^*) = \frac{M_N(\mu + \bar{\omega}, \mu + \omega)}{M_N(0,0)} \times \exp\left(-\frac{1}{2} \int_0^{\phi} \left[\tilde{h}\left(\cot\frac{\theta}{2}\right) + \omega_2(N + \omega_1 - \mu) + (\mu + \omega_1)^2 \cot\frac{\theta}{2} \right] \mathrm{d}\theta \right). \tag{1.39}$$

We remark that according to the theory in the sentence including (1.30)

$$\tilde{h}\left(\cot\frac{\theta}{2}\right) = -2\mathrm{i}\sigma_{\mathrm{VI}}\left(\frac{1}{1+\mathrm{e}^{\mathrm{i}\theta}}\right). \tag{1.40}$$

A generalization of the Jacobi ensemble (1.11) may also be made in a similar fashion, by the introduction of the additional factor $\prod_{l=1}^{N} |s - x_l|^{\mu}$ in the integrand. In the case $\xi = 0$ it is shown in [7] that this generalized ensemble can be related to (1.35), and consequently that

$$\sigma_{\mathrm{VI}}(t) = (e_2[-v_1, -v_2, v_3] + N\mu)\, t - \tfrac{1}{2} e_2[-v_1, -v_2, v_3, v_4] - N\mu + t(t-1)\frac{\mathrm{d}}{\mathrm{d}t} \log A_N(t; \omega_1, \omega_2, \mu; \xi^* = 0) \tag{1.41}$$

is a solution of (1.3) with the parameters

$$v_1 = \frac{N + \omega - \mu}{2}, \quad v_2 = \bar{\omega} + \frac{N + \omega + \mu}{2}, \quad v_3 = \frac{N - \omega + \mu}{2}, \quad v_4 = -\mu - \frac{N + \omega + \mu}{2}. \tag{1.42}$$

Moreover, a derivation of the transformation implied by (1.40) and (1.41) was given which tells us that the condition $\xi^* = 0$ in (1.41) can be relaxed.

The above-mentioned generalization of the Jacobi ensemble (1.11) has also revealed different Painlevé connections to those found in [7]. In the mid-1990s, with $\xi = 0$, this was studied from the viewpoint of orthogonal polynomial theory by Magnus [18]. In that study an auxiliary quantity occurring in the theory was shown to satisfy the $\mathrm{P_{VI}}$ equation (1.2) for appropriate parameters. We remark that further development of orthogonal polynomial theory in relation to (1.35) [8, 10] has been shown to relate to Bäcklund transformations in the Hamiltonian theory of $\mathrm{P_{VI}}$, and to the so-called discrete $\mathrm{dP_V}$ equations.

In the case $\omega_2 = \mu = 0, \omega_1 = \eta$ (1.35) is the generating function for the probability that there are exactly n eigenvalues in $(\pi - \phi, \pi)$ for the eigenvalue pdf on the right-hand side of (1.35). In this case expanding (1.14) allows us to specify the $s \to \infty$ boundary condition which must be satisfied by $\tilde{h}(s)$. However, for generic parameters the boundary conditions were not given in [7]. Consideration of this latter problem is our concern for the remainder of the paper. A similar situation was recently rectified in [9] in relation to a class of multidimensional integral solutions of $\sigma\mathrm{P_V}$ [6]. The approach taken was to write the multidimensional integral as a determinant (involving the confluent hypergeometric function), and to expand the entries of the determinant. An analogous strategy suffices in relation to (1.35), now with the entries of the determinant given in terms of the Gauss hypergeometric function. This is done in section 2.

The form of the general expansion of a solution of the $\sigma\mathrm{P_{VI}}$ equation (1.3), or equivalently its τ function about its fixed singularities $t = 0, 1, \infty$ has been given by Jimbo [15]. The exponents occurring therein are given in terms of the monodromy data associated with the isomonodromy deformation formulation of $\mathrm{P_{VI}}$. Aspects of this theory and the results of Jimbo are revised in section 3.

One of the features of the general expansions is that they consist of two branches, whereas the expansion of (1.35) only exhibits a single branch. We show, in section 4, that this is entirely consistent with the monodromy data associated with σ_{VI}; it is such that the coefficient in front of one of the branches vanishes identically.

2. Boundary conditions

Consider the multidimensional integral (1.35). According to (1.40) (with the condition $\xi^* = 0$ relaxed), we know how to relate its logarithmic derivative to a solution of the $\sigma\mathrm{P_{VI}}$ equation (1.3). However, this property cannot be used to characterize the multidimensional integral unless an appropriate boundary condition is specified. As remarked in the second paragraph below (1.42) above, in [7] the interpretation of (1.35) for some special parameters as the generating function of a gap probability provided the boundary condition in those cases. However, for general parameters no boundary condition was presented.

Taking a different approach, namely the expansion of the elements in the determinant form of (1.35), the sought boundary conditions can be deduced as presented in the following result.

Proposition 2.1. *For generic values of $\mu, \omega, \bar\omega$ the spectral average A_N has the following expansions. About $t = 0$ subject to $\mu - \bar\omega \notin \mathbb{Z}$, we have*

$$t^{N\mu} A_N \underset{t\to 0}{\sim} \left\{1 + \xi^* \frac{\mathrm{e}^{-\pi\mathrm{i}(\mu-\bar\omega)}}{2\mathrm{i}\sin\pi(\mu-\bar\omega)}\right\}^N \prod_{k=0}^{N-1} \frac{k!\Gamma(2\omega_1+k+1)}{\Gamma(1+k+\mu+\omega)\Gamma(1+k-\mu+\bar\omega)}$$
$$\times \left\{1 + \frac{2N\mu(\mu+\omega)}{N-\mu+\bar\omega} t - \frac{\xi^*\mathrm{e}^{-\pi\mathrm{i}(\mu-\bar\omega)}}{2\mathrm{i}\sin\pi(\mu-\bar\omega) + \xi^*\mathrm{e}^{-\pi\mathrm{i}(\mu-\bar\omega)}}\right.$$
$$\left.\times \frac{\Gamma(1+\mu+\omega)\Gamma(1+\mu-\bar\omega)\Gamma(1+2\mu)\Gamma(N-\mu+\bar\omega)}{\Gamma(N)\Gamma(N+\mu+\bar\omega)\Gamma(N+2\omega_1)\Gamma^2(2-N+\mu-\bar\omega)} t^{1-N+\mu-\bar\omega}\right\}. \tag{2.1}$$

About $t = 1$ subject to $2\mu + 2\omega_1 \notin \mathbb{Z}$, we have

$$A_N \underset{t\to 1}{\sim} \prod_{k=0}^{N-1} \frac{k!\Gamma(2\mu+2\omega_1+k+1)}{\Gamma(1+k+\mu+\omega)\Gamma(1+k+\mu+\bar\omega)} \left\{1 + \frac{N\mu(\bar\omega-\omega)}{2\mu+2\omega_1}(1-t)\right.$$
$$+ \frac{(-1)^{N+1}}{\sin\pi(2\mu+2\omega_1)} \left(\xi^* \frac{\mathrm{e}^{-\pi\mathrm{i}(\mu-\bar\omega)}}{2\mathrm{i}} + \frac{\sin\pi 2\mu \sin\pi(\mu+\omega)}{\sin\pi(2\mu+2\omega_1)}\right)$$
$$\left.\times \frac{\Gamma(1+2\mu)\Gamma(1+2\omega_1)\Gamma(1+\mu+\omega)\Gamma(1+\mu+\bar\omega)}{\Gamma^2(2\mu+2\omega_1+2)\Gamma(2\mu+2\omega_1+1)\Gamma(N)\Gamma(-N-2\mu-2\omega_1)} (1-t)^{1+2\mu+2\omega_1}\right\}. \tag{2.2}$$

And about $t = \infty$ subject to $\mu - \omega \notin \mathbb{Z}$, we have

$$t^{-\mu N} A_N \underset{t\to\infty}{\sim} \left\{ -\frac{e^{-\pi i 2\mu}}{\sin\pi(\mu-\omega)} \left(\sin\pi(\mu+\omega) + \xi^* \frac{e^{-\pi i(\mu+\omega)}}{2i} \right) \right\}^N \\ \times \prod_{k=0}^{N-1} \frac{k!\Gamma(2\omega_1+k+1)}{\Gamma(1+k+\mu+\bar{\omega})\Gamma(1+k-\mu+\omega)} \left\{ 1 + \frac{2\mu N(\mu+\bar{\omega})}{N-\mu+\omega}\frac{1}{t} \right. \\ + e^{\pi i(\mu-\omega)} \left(\frac{2i\sin\pi 2\mu + \xi^* e^{-\pi i 2\mu}}{2i\sin\pi(\mu+\omega) + \xi^* e^{-\pi i(\mu+\omega)}} \right) \\ \left. \times \frac{\Gamma(1+2\mu)\Gamma(1+\mu+\bar{\omega})\Gamma(1+\mu-\omega)\Gamma(N-\mu+\omega)}{\Gamma(N)\Gamma(N+\mu+\omega)\Gamma(N+2\omega_1)\Gamma^2(2-N+\mu-\omega)} t^{-1+N-\mu+\omega} \right\}. \tag{2.3}$$

Any one of these boundary conditions suffices to uniquely define a solution to the ordinary differential equation (1.3) under the above generic restrictions on the parameters.

Proof. Using Heine's identity we can deduce from (1.35) that

$$A_N(t;\omega_1,\omega_2,\mu;\xi^*) \\ = t^{-N\mu} \det\left[\left(\int_{-\pi}^{\pi} - \xi^* \int_{\pi-\phi}^{\pi} \right) \frac{d\theta}{2\pi} z^{-\mu-\omega+j-k}(1+z)^{2\omega_1}(1+tz)^{2\mu} \right]_{0\leqslant j,k\leqslant N-1}. \tag{2.4}$$

This is a determinant of a Toeplitz matrix whose symbol, or more specifically weight $w(z)$, is one defined on the unit circle $|z| = 1$ by

$$w(z) = t^{-\mu} z^{-\mu-\omega}(1+z)^{2\omega_1}(1+tz)^{2\mu} \begin{cases} 1, & \theta \in (-\pi, \pi-\phi) \\ 1-\xi^*, & \theta \in (\pi-\phi, \pi), \end{cases} \tag{2.5}$$

with Fourier components w_k such that $w(z) = \sum_{k=-\infty}^{\infty} w_k z^k$. Considering the expansion about $t = 0$ first we note that the Toeplitz matrix element can be evaluated as

$$t^{\mu} w_n = \left\{ 1 + \xi^* \frac{e^{-\pi i(n+\mu-\bar{\omega})}}{2i\sin\pi(n+\mu-\bar{\omega})} \right\} \frac{\Gamma(2\omega_1+1)}{\Gamma(1+n+\mu+\omega)\Gamma(1-n-\mu+\bar{\omega})} \\ \times {}_2F_1(-2\mu, -n-\mu-\omega; 1-n-\mu+\bar{\omega}; t) \\ - \xi^* \frac{e^{-\pi i(n+\mu-\bar{\omega})}}{2i\sin\pi(n+\mu-\bar{\omega})} \frac{\Gamma(2\mu+1)}{\Gamma(1+n+\mu-\bar{\omega})\Gamma(1-n+\mu+\bar{\omega})} \\ \times t^{n+\mu-\bar{\omega}} {}_2F_1(-2\omega_1, n-\mu-\bar{\omega}; 1+n+\mu-\bar{\omega}; t), \tag{2.6}$$

under the condition that $n + \mu - \bar{\omega} \notin \mathbb{Z}$, which is a form suitable for the development of an expansion about $t = 0$. In the course of the derivation we had to invoke tighter constraints on the parameters, namely $|t| = 1, t \neq 1$ and $\Re(2\mu) \geqslant 0, \Re(2\omega_1) \geqslant 0$, but these can be relaxed by analytic continuation arguments. The structure is $t^{\mu} w_n = a_n(t) + t^{n+\mu-\bar{\omega}} b_n(t)$, where $a_n(t), b_n(t)$ are analytic about $t = 0$. One can expand the Toeplitz determinant about $t = 0$ retaining only the leading order terms from both the analytic and non-analytic contributions and the resulting formula is

$$t^{\mu N} A_N(t) \underset{t\to 0}{\sim} \det(a_{j-k}(0) + t a'_{j-k}(0))_{j,k=0,\ldots,N-1} \\ + (-1)^{N-1} b_{-(N-1)} \det(a_{j-k+1}(0))_{j,k=0,\ldots,N-2} t^{\mu-\bar{\omega}-(N-1)}. \tag{2.7}$$

Using the determinant identity [21]

$$\det\left(\frac{\Gamma(d+k-j)}{\Gamma(c+k-j)} \right)_{0\leqslant j,k\leqslant n-1} = \prod_{j=0}^{n-1} j! \frac{\Gamma(1+d-c)}{\Gamma(1+d-c-j)} \frac{\Gamma(d-n+1+j)}{\Gamma(c+j)}, \tag{2.8}$$

or its slightly more general form

$$\det\left(\frac{\Gamma(z_k+b-j)}{\Gamma(z_k-j)}\right)_{0\leqslant j,k\leqslant n-1} = \prod_{0\leqslant j<k\leqslant n-1}(z_k-z_j)\prod_{j=0}^{n-1}(-1)^j(-b)_j\frac{\Gamma(z_j+b-n+1)}{\Gamma(z_j)}, \tag{2.9}$$

for an arbitrary sequence $\{z_j\}_{j=0}^{n-1}$ one can evaluate the determinants appearing in (2.7). The result is (2.1).

To develop the expansion about $t=1$ the appropriate expression for the Toeplitz matrix element is

$$\begin{aligned} t^{\bar{\omega}-n}w_n = {} & \frac{\Gamma(2\mu+2\omega_1+1)}{\Gamma(1+n+\mu+\omega)\Gamma(1-n+\mu+\bar{\omega})}\,{}_2F_1(-2\omega_1, n-\mu-\bar{\omega}; -2\mu-2\omega_1; 1-t) \\ & + \frac{1}{\pi}\left(\xi^*\frac{\mathrm{e}^{-\pi\mathrm{i}(n+\mu-\bar{\omega})}}{2\mathrm{i}} + \frac{\sin\pi 2\mu\sin\pi(n+\mu+\omega)}{\sin\pi(2\mu+2\omega_1)}\right)\frac{\Gamma(1+2\mu)\Gamma(1+2\omega_1)}{\Gamma(2+2\mu+2\omega_1)} \\ & \times(1-t)^{1+2\mu+2\omega_1}\,{}_2F_1(1+2\mu, n+1+\mu+\omega; 2+2\mu+2\omega_1; 1-t), \end{aligned} \tag{2.10}$$

subject to $1+2\mu+2\omega_1 \notin \mathbb{Z}$. The structure is now $t^{\bar{\omega}-n}w_n = a_n(1-t) + (1-t)^{1+2\mu+2\omega_1}b_n(1-t)$, where $a_n(1-t), b_n(1-t)$ are analytic about $t=1$. Again one can expand the Toeplitz determinant about $t=1$ retaining only the leading order terms from both the analytic and non-analytic contributions and use the above determinant identities to arrive at (2.2).

Lastly the expansion about $t=\infty$ is computed using the Toeplitz matrix element in the form

$$\begin{aligned} t^{\mu}w_n = {} & \frac{\mathrm{e}^{-\pi\mathrm{i}2\mu}}{\sin\pi(n-\mu+\omega)}\left(\sin\pi(n+\mu+\omega)+\xi^*\frac{\mathrm{e}^{-\pi\mathrm{i}(n+\mu+\omega)}}{2\mathrm{i}}\right) \\ & \times\frac{\Gamma(2\omega_1+1)}{\Gamma(1+n-\mu+\omega)\Gamma(1-n+\mu+\bar{\omega})}t^{2\mu}\,{}_2F_1(-2\mu, n-\mu-\bar{\omega}; 1+n-\mu+\omega; 1/t) \\ & - \frac{\mathrm{e}^{-\pi\mathrm{i}(n+\mu+\omega)}}{\sin\pi(n-\mu+\omega)}\left(\sin\pi 2\mu+\xi^*\frac{\mathrm{e}^{-\pi\mathrm{i}2\mu}}{2\mathrm{i}}\right)\frac{\Gamma(2\mu+1)}{\Gamma(1+n+\mu+\omega)\Gamma(1-n+\mu-\omega)} \\ & \times t^{n+\mu+\omega}\,{}_2F_1(-2\omega_1, -n-\mu-\omega; 1-n+\mu-\omega; 1/t), \end{aligned} \tag{2.11}$$

valid for $n-\mu+\omega \notin \mathbb{Z}$. The structure is $t^{-\mu}w_n = a_n(1/t) + t^{n-\mu+\omega}b_n(1/t)$, where $a_n(1/t), b_n(1/t)$ are analytic about $t=\infty$. Repeating the procedure adopted about the other singularities one arrives at (2.3). □

Remark 2.1. We do not attempt to treat the degenerate cases where either $\mu-\bar{\omega}\in\mathbb{Z}, 2\mu+2\omega_1\in\mathbb{Z}$ or $\mu-\omega\in\mathbb{Z}$ here as this results in confluent logarithms and other technical difficulties.

3. Isomonodromy deformation formulation for P_{VI}

In this section we describe the isomonodromic deformation system which characterizes the general solution to the sixth Painlevé equation and outline a solution to the direct monodromy problem, that is, explicit formulae for the monodromy data associated with a particular solution of the σ-form for the sixth Painlevé equation. The isomonodromic deformation system associated with the sixth Painlevé equation is not uniquely determined and we shall see this arbitrariness arise in section 4 when applying the general theory to our random matrix theory.

Following the conventions and notations of [15, 17], we consider the 2×2 linear matrix ODE for $\Psi(\lambda; t)$ with four regular singularities in the λ-plane chosen to be $\nu = 0, t, 1, \infty$

$$\frac{\partial}{\partial \lambda}\Psi = \left(\frac{A_0}{\lambda} + \frac{A_1}{\lambda - 1} + \frac{A_t}{\lambda - t}\right)\Psi, \tag{3.1}$$

$$\frac{\partial}{\partial t}\Psi = -\frac{A_t}{\lambda - t}\Psi. \tag{3.2}$$

It is taken that the residue matrices $A_\nu(t)$ satisfy

$$A_0 + A_t + A_1 = -A_\infty = -\frac{\theta_\infty}{2}\sigma_3, \qquad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \theta_\infty \in \mathbb{C}\backslash\mathbb{Z}, \tag{3.3}$$

and given the freedom to choose either tr A_ν or one of the eigenvalues we follow the convention of Jimbo [15],

$$\operatorname{tr} A_\nu = 0, \qquad \det A_\nu = -\tfrac{1}{4}\theta_\nu^2, \qquad \nu = 0, t, 1, \infty, \tag{3.4}$$

defining the formal exponents of monodromy θ_ν. The residue matrices satisfy the *Schlesinger system* of equations

$$\frac{\mathrm{d}}{\mathrm{d}t}A_0 = -\frac{[A_0, A_t]}{t}, \tag{3.5}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}A_t = \frac{[A_0, A_t]}{t} + \frac{[A_t, A_1]}{1 - t}, \tag{3.6}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}A_1 = -\frac{[A_t, A_1]}{1 - t}, \tag{3.7}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}A_\infty = 0 \tag{3.8}$$

as a consequence of the compatibility of (3.1) and (3.2).

The τ*-function* for $\mathrm{P_{VI}}$ is defined by

$$\frac{\mathrm{d}}{\mathrm{d}t}\log\tau = \operatorname{tr}\left(\frac{A_0}{t} + \frac{A_1}{t - 1}\right)A_t \tag{3.9}$$

and the σ*-function*

$$\zeta(t) = t(t-1)\frac{\mathrm{d}}{\mathrm{d}t}\log\tau + \frac{1}{4}\left(\theta_t^2 - \theta_\infty^2\right)t - \frac{1}{8}\left(\theta_t^2 + \theta_0^2 - \theta_\infty^2 - \theta_1^2\right) \tag{3.10}$$

which satisfies the second-order second-degree differential equation

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\zeta\left(t(t-1)\frac{\mathrm{d}^2}{\mathrm{d}t^2}\zeta\right)^2 &+ \left[2\frac{\mathrm{d}}{\mathrm{d}t}\zeta\left(t\frac{\mathrm{d}}{\mathrm{d}t}\zeta - \zeta\right) - \left(\frac{\mathrm{d}}{\mathrm{d}t}\zeta\right)^2 - \frac{1}{16}\left(\theta_t^2 - \theta_\infty^2\right)\left(\theta_0^2 - \theta_1^2\right)\right]^2 \\
&= \left(\frac{\mathrm{d}}{\mathrm{d}t}\zeta + \frac{1}{4}(\theta_t + \theta_\infty)^2\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}\zeta + \frac{1}{4}(\theta_t - \theta_\infty)^2\right) \\
&\quad \times \left(\frac{\mathrm{d}}{\mathrm{d}t}\zeta + \frac{1}{4}(\theta_0 + \theta_1)^2\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}\zeta + \frac{1}{4}(\theta_0 - \theta_1)^2\right),
\end{aligned} \tag{3.11}$$

(cf (1.3)).

Furthermore, we suppose (assumption 1) that the matrices A_ν are diagonalizable, i.e., that there exists non-singular $R_\nu \in SL(2, \mathbb{C})$ such that

$$R_\nu^{-1}A_\nu R_\nu = \tfrac{1}{2}\theta_\nu\sigma_3, \qquad \theta_\nu \in \mathbb{C}\backslash\mathbb{Z}. \tag{3.12}$$

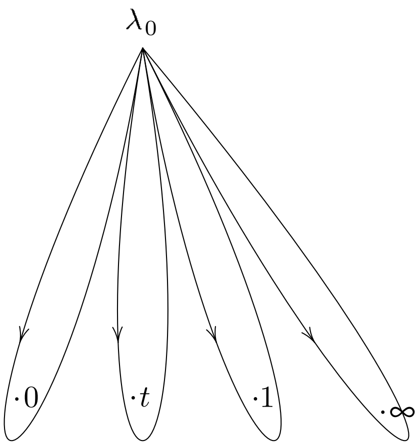

Figure 1. Monodromy representation of the fundamental group for $\mathbb{CP}\backslash\{0, t, 1, \infty\}$.

In the neighbourhood of a regular singularity $\Psi(\lambda)$ can be expanded locally as

$$\Psi = \sum_{m=0}^{\infty} \Psi_{m\nu}(\lambda - \nu)^{m+\frac{\theta_\nu}{2}\sigma_3} C_\nu, \tag{3.13}$$

for $\nu = 0, t, 1$ and for $\lambda = \infty$ in the form

$$\Psi = \left(I + \sum_{m=1}^{\infty} \Psi_{m\infty}\lambda^{-m} \right) \lambda^{-\frac{\theta_\infty}{2}\sigma_3}. \tag{3.14}$$

For such local series to exist we have to assume (assumption 2) that the eigenvalues of A_ν are distinct modulo the non-zero integers, i.e. that $\pm\theta_\nu \notin \mathbb{N}$.

The matrices $C_\nu, \nu = 0, t, 1$ are the connection matrices and we are taking the local solutions (3.14) as our fundamental system of solutions, i.e. $C_\infty = I$. The monodromy matrices $M_\nu(\nu = 0, t, 1, \infty)$ are defined as

$$\Psi|_{\lambda=\nu+\delta\,\mathrm{e}^{2\pi\mathrm{i}}} = \Psi|_{\lambda=\nu+\delta} M_\nu, \qquad \nu = 0, t, 1, \infty, \tag{3.15}$$

and are given in terms of the monodromy exponents and connection matrices by

$$M_\nu = C_\nu^{-1}\,\mathrm{e}^{\pi\mathrm{i}\theta_\nu\sigma_3} C_\nu. \tag{3.16}$$

The monodromy matrices together satisfy the cyclic relation

$$M_\infty M_1 M_t M_0 = I, \tag{3.17}$$

according to the convention taken for the basis of loops displayed in figure 1 which generate the fundamental group $\pi(\mathbb{CP}\backslash\{0, t, 1, \infty\}, \lambda_0)$. There is arbitrariness in the monodromy data in the sense that the replacement $C_\nu \mapsto D_\nu^{-1}C_\nu$ does not change the monodromy matrices provided that D_ν commutes with the right-hand side of (3.12). This implies that D_ν is diagonal if $\theta_\nu \neq 0$. This arbitrariness will manifest itself in the appearance of an arbitrary complex number in the explicit parameterization discussed later.

The isomonodromic principle states that the monodromy data $MD := \{\theta_\nu, C_\nu, M_\nu | \nu = 0, t, 1, \infty\}$ are preserved under the deformations of t. The invariants of the monodromy data are defined,

$$p_\mu = 2\cos\pi\theta_\mu := \mathrm{tr}\ M_\mu \qquad \mu \in \{0, t, 1, \infty\} \tag{3.18}$$

$$p_{\mu\nu} = 2\cos\pi\sigma_{\mu\nu} := \operatorname{tr} M_\mu M_\nu \qquad \mu, \nu \in \{0, t, 1\}, \tag{3.19}$$

in the sense that these do not contain any arbitrary constants.

In [15] Jimbo states the following conditions under which his results apply:

$$\theta_0, \theta_t, \theta_1, \theta_\infty \notin \mathbb{Z}, \tag{3.20}$$

$$0 < \Re(\sigma_{0t}) < 1, \tag{3.21}$$

$$\theta_0 \pm \theta_t \pm \sigma_{0t}, \qquad \theta_\infty \pm \theta_1 \pm \sigma_{0t} \notin 2\mathbb{Z}. \tag{3.22}$$

When $\sigma_{0t} \neq 0$ a parameterization of the monodromy matrices was deduced by Jimbo and is given in lemma 3.1.

Lemma 3.1 (Jimbo [15]). *Subject to conditions (3.20), (3.21) and (3.22) the monodromy matrices can be parameterized in the following way:*

$$M_\infty = \begin{pmatrix} e^{\pi i\theta_\infty} & 0 \\ 0 & e^{-\pi i\theta_\infty} \end{pmatrix}, \tag{3.23}$$

$$M_1 = \frac{1}{i\sin\pi\theta_\infty}\begin{pmatrix} \cos\pi\sigma - e^{-\pi i\theta_\infty}\cos\pi\theta_1 & -2r\, e^{-\pi i\theta_\infty}\sin\frac{\pi}{2}(\theta_\infty+\theta_1+\sigma)\sin\frac{\pi}{2}(\theta_\infty+\theta_1-\sigma) \\ 2r^{-1}\, e^{\pi i\theta_\infty}\sin\frac{\pi}{2}(\theta_\infty-\theta_1+\sigma)\sin\frac{\pi}{2}(\theta_\infty-\theta_1-\sigma) & -\cos\pi\sigma + e^{\pi i\theta_\infty}\cos\pi\theta_1 \end{pmatrix}, \tag{3.24}$$

$$CM_tC^{-1} = \frac{1}{i\sin\pi\sigma}\begin{pmatrix} e^{\pi i\sigma}\cos\pi\theta_t - \cos\pi\theta_0 & -2s\, e^{\pi i\sigma}\sin\frac{\pi}{2}(\theta_0+\theta_t-\sigma)\sin\frac{\pi}{2}(\theta_0-\theta_t+\sigma) \\ 2s^{-1}\, e^{-\pi i\sigma}\sin\frac{\pi}{2}(\theta_0+\theta_t+\sigma)\sin\frac{\pi}{2}(\theta_0-\theta_t-\sigma) & -e^{-\pi i\sigma}\cos\pi\theta_t + \cos\pi\theta_0 \end{pmatrix}, \tag{3.25}$$

$$CM_0C^{-1} = \frac{1}{i\sin\pi\sigma}\begin{pmatrix} e^{\pi i\sigma}\cos\pi\theta_0 - \cos\pi\theta_t & 2s\sin\frac{\pi}{2}(\theta_0+\theta_t-\sigma)\sin\frac{\pi}{2}(\theta_0-\theta_t+\sigma) \\ -2s^{-1}\sin\frac{\pi}{2}(\theta_0-\theta_t-\sigma)\sin\frac{\pi}{2}(\theta_0+\theta_t+\sigma) & -e^{-\pi i\sigma}\cos\pi\theta_0 + \cos\pi\theta_t \end{pmatrix}, \tag{3.26}$$

where

$$C = \begin{pmatrix} \sin\frac{\pi}{2}(\theta_\infty-\theta_1-\sigma) & r\sin\frac{\pi}{2}(\theta_\infty+\theta_1+\sigma) \\ r^{-1}\sin\frac{\pi}{2}(\theta_\infty-\theta_1+\sigma) & \sin\frac{\pi}{2}(\theta_\infty+\theta_1-\sigma) \end{pmatrix}. \tag{3.27}$$

Here r is an arbitrary non-zero complex number, and the short-hand notation $s := s_{0t}$, $\sigma = \sigma_{0t}$ is used.

Proof. We begin by noting that (3.23) follows from our choice for the fundamental system of solutions. To establish the other formula we require two preliminary results. Let us make the abbreviations $c_\nu := \cos\pi\theta_\nu$, $s_\nu := \sin\pi\theta_\nu$, $c_\sigma := \cos\pi\sigma_{0t}$, $s_\sigma := \sin\pi\sigma_{0t}$, $\epsilon_\nu := c_\nu + is_\nu$ and

$$\mathfrak{S}(\vartheta) := \sin\frac{\pi}{2}\vartheta, \tag{3.28}$$

and note some trigonometric identities which will be useful in this and the ensuing proofs

$$2\mathfrak{S}(\theta_\infty - \theta_1 + \sigma)\mathfrak{S}(\theta_\infty + \theta_1 + \sigma) = c_1 - c_\infty c_\sigma + s_\infty s_\sigma, \tag{3.29}$$

$$2\mathfrak{S}(\theta_\infty + \theta_1 - \sigma)\mathfrak{S}(\theta_\infty - \theta_1 - \sigma) = c_1 - c_\infty c_\sigma - s_\infty s_\sigma, \tag{3.30}$$

$$2\mathfrak{S}(\theta_\infty - \theta_1 + \sigma)\mathfrak{S}(\theta_\infty - \theta_1 - \sigma) = c_\sigma - c_1 c_\infty - s_1 s_\infty, \tag{3.31}$$

$$2\mathfrak{S}(\theta_\infty + \theta_1 + \sigma)\mathfrak{S}(\theta_\infty + \theta_1 - \sigma) = c_\sigma - c_1 c_\infty + s_1 s_\infty, \tag{3.32}$$

$$2\mathfrak{S}(\theta_\infty - \theta_1 + \sigma)\mathfrak{S}(\theta_\infty + \theta_1 - \sigma) = -c_\infty + c_1 c_\sigma + s_1 s_\sigma, \tag{3.33}$$

$$2\mathfrak{S}(\theta_\infty + \theta_1 + \sigma)\mathfrak{S}(\theta_\infty - \theta_1 - \sigma) = -c_\infty + c_1 c_\sigma - s_1 s_\sigma, \tag{3.34}$$

$$2\mathfrak{S}(\theta_0 - \theta_t + \sigma)\mathfrak{S}(\theta_0 + \theta_t + \sigma) = c_t - c_0 c_\sigma + s_0 s_\sigma, \tag{3.35}$$

$$2\mathfrak{S}(\theta_0 + \theta_t - \sigma)\mathfrak{S}(\theta_0 - \theta_t - \sigma) = c_t - c_0 c_\sigma - s_0 s_\sigma, \tag{3.36}$$

$$2\mathfrak{S}(\theta_0 - \theta_t + \sigma)\mathfrak{S}(\theta_0 + \theta_t - \sigma) = -c_0 + c_t c_\sigma + s_t s_\sigma, \tag{3.37}$$

$$2\mathfrak{S}(\theta_0 + \theta_t + \sigma)\mathfrak{S}(\theta_0 - \theta_t - \sigma) = -c_0 + c_t c_\sigma - s_t s_\sigma. \tag{3.38}$$

We also note that a general member M of $\mathrm{SL}(2,\mathbb{C})$, characterized by $\det M = 1$ and $\operatorname{tr} M = 2c$, can be written as

$$M = \begin{pmatrix} c + x & -(s + \mathrm{i}x)b \\ \dfrac{s - \mathrm{i}x}{b} & c - x \end{pmatrix}, \tag{3.39}$$

where $c^2 + s^2 = 1$, and with arbitrary complex numbers $b, b^{-1} \neq 0, x^{-1} \neq 0$. One can parameterize $c = \cos\pi\theta, s = \sin\pi\theta$. The observation made in [2] that simplifies the derivation is that the Jimbo parameterization is constructed so that

$$C M_t M_0 C^{-1} = \Delta := \begin{pmatrix} \epsilon_\sigma & 0 \\ 0 & \epsilon_\sigma^{-1} \end{pmatrix}, \tag{3.40}$$

for a non-singular matrix $C \in \mathrm{GL}(2,\mathbb{C})$. This implies

$$C M_1^{-1} M_\infty^{-1} C^{-1} = \begin{pmatrix} \epsilon_\sigma & 0 \\ 0 & \epsilon_\sigma^{-1} \end{pmatrix}. \tag{3.41}$$

Using (3.23) and the parameterization (3.39) for M_1 we can recast the above relation as a homogeneous matrix equation for C, that is,

$$\begin{pmatrix} \epsilon_\sigma & 0 \\ 0 & \epsilon_\sigma^{-1} \end{pmatrix} C = C \begin{pmatrix} \epsilon_\infty(c_1 + x) & -\epsilon_\infty(s_1 + \mathrm{i}x)b \\ \dfrac{s_1 - \mathrm{i}x}{b\epsilon_\infty} & \dfrac{c_1 - x}{\epsilon_\infty} \end{pmatrix}. \tag{3.42}$$

This has a solution if and only if $\mathrm{i}x s_\infty = c_\sigma - c_1 c_\infty$, which fixes x. Using the identities (3.29), (3.30), (3.31), (3.32) the equation for the components of C can be written as

$$-\mathrm{i}\epsilon_\infty b\mathfrak{S}(\theta_\infty + \theta_1 + \sigma)C_{11} + \mathfrak{S}(\theta_\infty - \theta_1 - \sigma)C_{12} = 0, \tag{3.43}$$

$$\epsilon_\infty b\mathfrak{S}(\theta_\infty + \theta_1 - \sigma)C_{21} + i\mathfrak{S}(\theta_\infty - \theta_1 + \sigma)C_{22} = 0, \tag{3.44}$$

assuming $\theta_\infty \pm \theta_1 \pm \sigma \neq \mathbb{Z}$. If one defines an arbitrary complex constant $r := \mathrm{i}b\epsilon_\infty$, then this yields the solution (3.27) for C. We note that $\det C = -s_\infty s_\sigma$. Employing the solution for x and the identities (3.31), (3.32) we arrive at formula (3.24) for M_1. For the remaining monodromy matrices we utilize the representation (3.39) for CM_0C^{-1} and CM_tC^{-1} in the relation (3.40). From the (1, 1) and (2, 2) components of the resulting matrix equation we find

$$\epsilon_\infty c_0 - c_t = \epsilon_\infty x_0 + x_t, \tag{3.45}$$

$$-\epsilon_\infty^{-1} c_0 + c_t = \epsilon_\infty^{-1} x_0 + x_t, \tag{3.46}$$

and one deduces that $\mathrm{i}s_\sigma x_0 = c_0 c_\sigma - c_t$ and $\mathrm{i}s_\sigma x_t = c_t c_\sigma - c_0$. From the other components we find that $x_t^2 - x_0^2 = c_t^2 - c_0^2$ and the formula for the ratio of the other undetermined constants is given by

$$\frac{b_t}{b_0} = \epsilon_\sigma \frac{\mathfrak{S}(\theta_0 - \theta_t - \sigma)}{\mathfrak{S}(\theta_0 - \theta_t + \sigma)}, \tag{3.47}$$

where we have utilized (3.35), (3.36), (3.37) and (3.38). If we define the remaining undetermined constant $b_t := -\mathrm{i}\epsilon_\sigma s_{0t}$, then we recover the parameterizations (3.26), (3.25) for M_0, M_t. □

A key identity is the following connection relation which relates s_{0t}, σ_{0t} to σ_{t1} and σ_{01}.

Lemma 3.2 (Jimbo [15]). *One of the connection relations is*

$$\begin{aligned}
4s_{0t}^{\pm 1} \sin\frac{\pi}{2}(\theta_0 + \theta_t \mp \sigma_{0t}) \sin\frac{\pi}{2}(\theta_0 - \theta_t \pm \sigma_{0t}) \sin\frac{\pi}{2}(\theta_\infty + \theta_1 \mp \sigma_{0t}) \sin\frac{\pi}{2}(\theta_\infty - \theta_1 \pm \sigma_{0t}) \\
= \mathrm{e}^{\pm\pi\mathrm{i}\sigma_{0t}} (\pm\mathrm{i}\sin\pi\sigma_{0t}\cos\pi\sigma_{t1} - \cos\pi\theta_t\cos\pi\theta_\infty - \cos\pi\theta_0\cos\pi\theta_1) \\
\pm\mathrm{i}\sin\pi\sigma_{0t}\cos\pi\sigma_{01} + \cos\pi\theta_t\cos\pi\theta_1 + \cos\pi\theta_\infty\cos\pi\theta_0.
\end{aligned} \tag{3.48}$$

Proof. The proof of this has been detailed in Boalch [2] with a typographical correction to the original formula in [15]. So we content ourselves with a brief summary of the steps involved. After noting (3.40) it was found that the monodromy invariants are more manageable in the forms

$$p_{t1} = \operatorname{tr}\left(CM_\infty^{-1}C^{-1}\Delta^{-1}(CM_tC^{-1})\right), \tag{3.49}$$

$$p_{01} = \operatorname{tr}\left(CM_\infty^{-1}C^{-1}\Delta^{-1}(CM_0C^{-1})\right), \tag{3.50}$$

because they were linear in $1, s_{0t}, s_{0t}^{-1}$. Upon taking the combination (3.49) + ϵ_σ(3.50) in order to eliminate the s_{0t}^{-1} terms it was found that the resulting expression could be factorized through the use of the identities

$$\mathfrak{S}(\theta_\infty + \theta_1 - \sigma)\mathfrak{S}(\theta_\infty - \theta_1 - \sigma) - \mathfrak{S}(\theta_\infty - \theta_1 + \sigma)\mathfrak{S}(\theta_\infty + \theta_1 + \sigma) = -s_\infty s_\sigma, \tag{3.51}$$

$$\begin{aligned}
\epsilon_\infty^{-1}\mathfrak{S}(\theta_\infty + \theta_1 - \sigma)\mathfrak{S}(\theta_\infty - \theta_1 - \sigma) - \epsilon_\infty\mathfrak{S}(\theta_\infty - \theta_1 + \sigma)\mathfrak{S}(\theta_\infty + \theta_1 + \sigma) \\
= \mathrm{i}s_\infty(\epsilon_\sigma c_\infty - c_1),
\end{aligned} \tag{3.52}$$

$$\begin{aligned}
\epsilon_\infty\mathfrak{S}(\theta_\infty + \theta_1 - \sigma)\mathfrak{S}(\theta_\infty - \theta_1 - \sigma) - \epsilon_\infty^{-1}\mathfrak{S}(\theta_\infty - \theta_1 + \sigma)\mathfrak{S}(\theta_\infty + \theta_1 + \sigma) \\
= \mathrm{i}s_\infty\left(c_1 - \epsilon_\sigma^{-1}c_\infty\right),
\end{aligned} \tag{3.53}$$

enabling a factor of $2s_\infty s_\sigma$ to be cancelled out. This then yielded one of the desired formulae (3.48), whilst the other could be found from the other combination of (3.49) and (3.50). □

A consequence of this is a constraint on the monodromy invariants $\{p_{0t}, p_{t1}, p_{01}\}$ which is an algebraic variety defining a sub-manifold, the monodromy manifold, of $\mathbb{C}^3$.

Lemma 3.3 (Jimbo [15]). *The monodromy manifold is given by*

$$\begin{aligned}
\mathfrak{M}(p_{0t}, p_{t1}, p_{01}) := p_{0t}p_{t1}p_{01} + p_{0t}^2 + p_{t1}^2 + p_{01}^2 - (p_0p_t + p_1p_\infty)p_{0t} - (p_tp_1 + p_0p_\infty)p_{t1} \\
- (p_0p_1 + p_tp_\infty)p_{01} + p_0^2 + p_t^2 + p_1^2 + p_\infty^2 + p_0p_tp_1p_\infty - 4 = 0.
\end{aligned} \tag{3.54}$$

Proof. We multiply the upper and lower sign forms of the left-hand side of (3.48) to eliminate s_{0t} and then employ the identities (3.29), (3.30), (3.35) and (3.36) to replace the product of sines. Equating this to the corresponding product of the right-hand sides yields (3.54) as the only non-trivial factor. □

Remark 3.1. The above connection relation involves only the free parameters s_{0t} and the monodromy invariants $\sigma_{0t}, \sigma_{t1}, \sigma_{01}$. As we shall see immediately below, the arbitrary parameters s_{0t} and σ_{0t} appear in the expansion for the τ-function about $t = 0$, and there exist analogous pairs about $t = 1, \infty$. Correspondingly there exist two other forms of the connection relation (3.48) involving either the parameters s_{t1}, s_{01} and can be deduced directly from (3.48) by a simple substitution rule given at the end of theorem 3.2. However, both connection relations yield the same formula (3.54) for the variety defining the monodromy manifold.

Now we come to the fundamental result for the expansion of the τ-function in the neighbourhood of the fixed singularities of the sixth Painlevé system at $t = 0, 1, \infty$.

Theorem 3.1 (Jimbo [15]). *Under conditions (3.20), (3.21) and (3.22) we have the expansion of the τ-function as $t \to 0$ in the domain $\{t \in \mathbb{C} | 0 < |t| < \varepsilon, |\arg(t)| < \phi\}$ for all $\varepsilon > 0$ and any $\phi > 0$:*

$$\tau(t) \sim C t^{(\sigma^2-\theta_0^2-\theta_t^2)/4} \left\{ 1 + \frac{\left(\theta_0^2-\theta_t^2-\sigma^2\right)\left(\theta_\infty^2-\theta_1^2-\sigma^2\right)}{8\sigma^2} t \right.$$
$$- \hat{s} \frac{\left[\theta_0^2-(\theta_t-\sigma)^2\right]\left[\theta_\infty^2-(\theta_1-\sigma)^2\right]}{16\sigma^2(1+\sigma)^2} t^{1+\sigma}$$
$$\left. - \hat{s}^{-1} \frac{\left[\theta_0^2-(\theta_t+\sigma)^2\right]\left[\theta_\infty^2-(\theta_1+\sigma)^2\right]}{16\sigma^2(1-\sigma)^2} t^{1-\sigma} + \mathrm{O}(|t|^{2(1-\Re(\sigma))}) \right\}, \tag{3.55}$$

where $\sigma \neq 0$ and $\hat{s}$ are related to s through

$$\hat{s} = s \frac{\Gamma^2(1-\sigma)\Gamma\left(1+\frac{1}{2}(\theta_0+\theta_t+\sigma)\right)\Gamma\left(1+\frac{1}{2}(-\theta_0+\theta_t+\sigma)\right)}{\Gamma^2(1+\sigma)\Gamma\left(1+\frac{1}{2}(\theta_0+\theta_t-\sigma)\right)\Gamma\left(1+\frac{1}{2}(-\theta_0+\theta_t-\sigma)\right)}$$
$$\times \frac{\Gamma\left(1+\frac{1}{2}(\theta_\infty+\theta_1+\sigma)\right)\Gamma\left(1+\frac{1}{2}(-\theta_\infty+\theta_1+\sigma)\right)}{\Gamma\left(1+\frac{1}{2}(\theta_\infty+\theta_1-\sigma)\right)\Gamma\left(1+\frac{1}{2}(-\theta_\infty+\theta_1-\sigma)\right)}, \tag{3.56}$$

and we employ the short-hand notation $s = s_{0t}, \hat{s} = \hat{s}_{0t}$ and $\sigma = \sigma_{0t}$. The monodromy data defining the unique solution to the sixth Painlevé system are $\{\sigma_{0t}, s_{0t}\}$. Here C is an arbitrary constant.

Proof. The details of this proof are given in Jimbo [15] and so we do not repeat them here. Also Guzzetti has laid out some of the intermediate steps in the appendix of his work on the elliptic representations of the general Painlevé six equation [12]. □

The regular singularities $x = 0, t, 1, \infty$ play equivalent roles and can be exchanged under linear fractional or Möbius transformations. Consequently one can solve the connection problem very neatly and under the additional conditions

$$0 < \Re(\sigma_{t1}), \Re(\sigma_{01}) < 1, \tag{3.57}$$

$$\theta_1 \pm \theta_t \pm \sigma_{t1}, \qquad \theta_\infty \pm \theta_0 \pm \sigma_{t1} \notin 2\mathbb{Z}, \tag{3.58}$$

$$\theta_0 \pm \theta_1 \pm \sigma_{01}, \qquad \theta_\infty \pm \theta_t \pm \sigma_{01} \notin 2\mathbb{Z} \tag{3.59}$$

derive expansions about $t = 1, \infty$.

Theorem 3.2 (Jimbo [15]). *Under conditions (3.20), (3.57) and (3.58) we have the expansion of the τ-function as $t \to 1$,*

$$\tau(t) \sim C(1-t)^{(\sigma_{t1}^2-\theta_1^2-\theta_t^2)/4}\Bigg\{1 + \frac{\left(\theta_1^2-\theta_t^2-\sigma_{t1}^2\right)\left(\theta_\infty^2-\theta_0^2-\sigma_{t1}^2\right)}{8\sigma_{t1}^2}(1-t)$$
$$- \hat{s}_{t1}\frac{\left[\theta_1^2-(\theta_t-\sigma_{t1})^2\right]\left[\theta_\infty^2-(\theta_0-\sigma_{t1})^2\right]}{16\sigma_{t1}^2(1+\sigma_{t1})^2}(1-t)^{1+\sigma_{t1}}$$
$$- \hat{s}_{t1}^{-1}\frac{\left[\theta_1^2-(\theta_t+\sigma_{t1})^2\right]\left[\theta_\infty^2-(\theta_0+\sigma_{t1})^2\right]}{16\sigma_{t1}^2(1-\sigma_{t1})^2}(1-t)^{1-\sigma_{t1}}$$
$$+\mathrm{O}(|1-t|^{2(1-\Re(\sigma_{t1}))})\Bigg\}, \tag{3.60}$$

and as $t \to \infty$,

$$\tau(t) \sim Ct^{-(\sigma_{01}^2-\theta_\infty^2+\theta_t^2)/4}\Bigg\{1 + \frac{\left(\theta_\infty^2-\theta_t^2-\sigma_{01}^2\right)\left(\theta_0^2-\theta_1^2-\sigma_{01}^2\right)}{8\sigma_{01}^2}t^{-1}$$
$$- \hat{s}_{01}\frac{\left[\theta_\infty^2-(\theta_t-\sigma_{01})^2\right]\left[\theta_0^2-(\theta_1-\sigma_{01})^2\right]}{16\sigma_{01}^2(1+\sigma_{01})^2}t^{-1-\sigma_{01}}$$
$$- \hat{s}_{01}^{-1}\frac{\left[\theta_\infty^2-(\theta_t+\sigma_{01})^2\right]\left[\theta_0^2-(\theta_1+\sigma_{01})^2\right]}{16\sigma_{01}^2(1-\sigma_{01})^2}t^{-1+\sigma_{01}} + \mathrm{O}(|t|^{-2(1-\Re(\sigma_{01}))})\Bigg\}. \tag{3.61}$$

Here $\hat{s}_{t1}, \hat{s}_{01}$ are found by making the following substitutions in (3.56), (3.48) respectively

$$\hat{s} \to \hat{s}_{t1}, \quad s \to s_{t1}, \quad \theta_0 \leftrightarrow \theta_1, \quad \sigma \to \sigma_{t1}, \quad \sigma_{t1} \to \sigma_{0t}, \tag{3.62}$$

$$\hat{s} \to \hat{s}_{01}, \quad s \to s_{01}, \quad \theta_0 \leftrightarrow \theta_\infty, \quad \sigma \to \sigma_{01}, \quad \sigma_{01} \to \tilde{\sigma}_{01}, \tag{3.63}$$

with

$$\cos\pi\tilde{\sigma}_{01} = -\cos\pi\sigma_{0t} - 2\cos\pi\sigma_{01}\cos\pi\sigma_{t1} + 2(\cos\pi\theta_0\cos\pi\theta_t + \cos\pi\theta_\infty\cos\pi\theta_1). \tag{3.64}$$

The monodromy data defining the unique solution to the sixth Painlevé system are either $\{\sigma_{t1}, s_{t1}\}$ or $\{\sigma_{01}, s_{01}\}$.

4. Monodromy data for the spectrum singularity ensemble

The precise relationship between the spectrum singularity average $A_N(t;)$ and the isomonodromy theory of the sixth Painlevé system is given by the following result. Its validity relies on the conjecture that the expansions of Jimbo given in theorems 3.1 and 3.2 remain valid upon relaxation of the constraints (3.20), (3.21), (3.22) and (3.57), (3.58), (3.59), provided the former are well defined (i.e. do not then diverge).

Proposition 4.1. *For the spectrum singularity ensemble the associated isomonodromic system is not unique but the monodromy data for any of these systems fall into three generic cases. An example of each case is given below in cases (A), (B) and (C). The formal monodromy exponents can be taken to belong to either of three sets*

$$\text{case } A: \quad \theta_0 = -\mu - \omega, \quad \theta_t = N + 2\omega_1, \quad \theta_1 = N + 2\mu, \quad \theta_\infty = -\mu - \bar{\omega}, \tag{4.1}$$

case B: $\theta_0 = \mu - \bar{\omega}, \quad \theta_t = N, \quad \theta_1 = N + 2\mu + 2\omega_1, \quad \theta_\infty = \mu - \omega,$ (4.2)

case C: $\theta_0 = -2\omega_1, \quad \theta_t = N + \mu + \omega, \quad \theta_1 = N + \mu + \bar{\omega}, \quad \theta_\infty = 2\mu.$ (4.3)

The monodromy invariants for either case are

$$\sigma_{0t} = N - \mu + \bar{\omega}, \quad \sigma_{t1} = 2\mu + 2\omega_1, \quad \sigma_{01} = N - \mu + \omega. \tag{4.4}$$

In case A the monodromy coefficients are

$$s_{0t} = 1 + \frac{2\mathrm{i}\sin\pi(\mu - \bar{\omega})}{\xi^* \,\mathrm{e}^{-\pi\mathrm{i}(\mu-\bar{\omega})}}, \tag{4.5}$$

$$s_{t1} = 1 + \xi^* \frac{\mathrm{e}^{-\pi\mathrm{i}(\mu-\bar{\omega})}}{2\mathrm{i}} \frac{\sin\pi(2\mu + 2\omega_1)}{\sin\pi 2\mu \sin\pi(\mu + \omega)}, \tag{4.6}$$

$$s_{01} = -\frac{\xi^* - 1 + \mathrm{e}^{2\pi\mathrm{i}(\mu+\omega)}}{\xi^* - 1 + \mathrm{e}^{4\pi\mathrm{i}\mu}}. \tag{4.7}$$

All monodromy matrices are lower triangular

$$M_0 = \begin{pmatrix} \mathrm{e}^{-\pi\mathrm{i}(\mu+\omega)} & 0 \\ m_0 & \mathrm{e}^{\pi\mathrm{i}(\mu+\omega)} \end{pmatrix}, \tag{4.8}$$

$$M_t = \begin{pmatrix} \mathrm{e}^{\pi\mathrm{i}(N+2\omega_1)} & 0 \\ m_t & \mathrm{e}^{-\pi\mathrm{i}(N+2\omega_1)} \end{pmatrix}, \tag{4.9}$$

$$M_1 = \begin{pmatrix} \mathrm{e}^{\pi\mathrm{i}(N+2\mu)} & 0 \\ m_1 & \mathrm{e}^{-\pi\mathrm{i}(N+2\mu)} \end{pmatrix}, \tag{4.10}$$

where

$$m_0 = \frac{2\mathrm{i}}{\sin\pi(\mu - \bar{\omega})} \left\{ \frac{\sin\pi 2\omega_1 \sin\pi(\mu + \bar{\omega})}{s_{0t}} - \frac{\sin\pi 2\mu \sin\pi(\mu + \omega)}{r} \right\}, \tag{4.11}$$

$$m_t = \frac{2\mathrm{i}(-1)^N \sin\pi 2\omega_1}{\sin\pi(\mu - \bar{\omega})} \left\{ -\frac{\sin\pi(\mu + \bar{\omega})}{s_{0t}} \mathrm{e}^{\pi\mathrm{i}(\mu-\bar{\omega})} + \frac{\sin\pi 2\mu}{r} \right\}, \tag{4.12}$$

$$m_1 = -\frac{2\mathrm{i}(-1)^N \sin\pi 2\mu}{r} \mathrm{e}^{-\pi\mathrm{i}(\mu+\bar{\omega})}. \tag{4.13}$$

For case B the monodromy coefficients are

$$s_{0t} = 1 + \frac{2\mathrm{i}\sin\pi(\mu - \bar{\omega})}{\xi^* \,\mathrm{e}^{-\pi\mathrm{i}(\mu-\bar{\omega})}}, \tag{4.14}$$

$$s_{t1} \frac{\sin\pi 2\omega_1 \sin\pi(\mu + \bar{\omega})}{\sin\pi(2\mu + 2\omega_1)} = \frac{\sin\pi 2\mu \sin\pi(\mu + \omega)}{\sin\pi(2\mu + 2\omega_1)} + \xi^* \frac{\mathrm{e}^{-\pi\mathrm{i}(\mu-\bar{\omega})}}{2\mathrm{i}}, \tag{4.15}$$

$$s_{01} = -\frac{\xi^* - 1 + \mathrm{e}^{2\pi\mathrm{i}(\mu+\omega)}}{\xi^* - 1 + \mathrm{e}^{4\pi\mathrm{i}\mu}}. \tag{4.16}$$

One of monodromy matrices is proportional to the identity, the others are full

$$M_0 = \frac{\mathrm{i}}{\sin\pi(\mu - \omega)} \begin{pmatrix} \mathrm{e}^{-\pi\mathrm{i}(\mu-\omega)} \cos\pi(\mu - \bar{\omega}) - \cos\pi(2\mu + 2\omega_1) & -\frac{2}{r} \sin\pi(\mu + \omega) \sin\pi 2\omega_1 \\ 2r \sin\pi(\mu + \bar{\omega}) \sin\pi 2\mu & -\mathrm{e}^{\pi\mathrm{i}(\mu-\omega)} \cos\pi(\mu - \bar{\omega}) + \cos\pi(2\mu + 2\omega_1) \end{pmatrix}, \tag{4.17}$$

$$M_t = (-1)^N I, \tag{4.18}$$

$$M_1 = \frac{\mathrm{i}(-1)^N}{\sin\pi(\mu-\omega)} \begin{pmatrix} \mathrm{e}^{-\pi\mathrm{i}(\mu-\omega)}\cos\pi(2\mu+2\omega_1) - \cos\pi(\mu-\bar\omega) & 2r\,\mathrm{e}^{-\pi\mathrm{i}(\mu-\omega)}\sin\pi(\mu+\bar\omega)\sin\pi 2\mu \\ \frac{2}{r}\,\mathrm{e}^{\pi\mathrm{i}(\mu-\omega)}\sin\pi(\mu+\omega)\sin\pi 2\omega_1 & \cos\pi(\mu-\bar\omega) - \mathrm{e}^{\pi\mathrm{i}(\mu-\omega)}\cos\pi(2\mu+2\omega_1) \end{pmatrix}. \tag{4.19}$$

For case C the monodromy coefficients are

$$s_{0t} = 1 + \frac{2\mathrm{i}\sin\pi(\mu-\bar\omega)}{\xi^*\,\mathrm{e}^{-\pi\mathrm{i}(\mu-\bar\omega)}}, \tag{4.20}$$

$$s_{t1}\frac{\sin\pi 2\omega_1\sin\pi(\mu+\bar\omega)}{\sin\pi(2\mu+2\omega_1)} = \frac{\sin\pi 2\mu\sin\pi(\mu+\omega)}{\sin\pi(2\mu+2\omega_1)} + \xi^*\frac{\mathrm{e}^{-\pi\mathrm{i}(\mu-\bar\omega)}}{2\mathrm{i}}, \tag{4.21}$$

$$s_{01} = -\frac{\xi^* - 1 + \mathrm{e}^{2\pi\mathrm{i}(\mu+\omega)}}{\xi^* - 1 + \mathrm{e}^{4\pi\mathrm{i}\mu}}. \tag{4.22}$$

All monodromy matrices are upper triangular

$$M_0 = \begin{pmatrix} \mathrm{e}^{\pi\mathrm{i}2\omega_1} & m_0 \\ 0 & \mathrm{e}^{-\pi\mathrm{i}2\omega_1} \end{pmatrix}, \tag{4.23}$$

$$M_t = \begin{pmatrix} \mathrm{e}^{-\pi\mathrm{i}(N+\mu+\omega)} & m_t \\ 0 & \mathrm{e}^{\pi\mathrm{i}(N+\mu+\omega)} \end{pmatrix}, \tag{4.24}$$

$$M_1 = \begin{pmatrix} \mathrm{e}^{-\pi\mathrm{i}(N+\mu+\bar\omega)} & m_1 \\ 0 & \mathrm{e}^{\pi\mathrm{i}(N+\mu+\bar\omega)} \end{pmatrix}, \tag{4.25}$$

where

$$m_0 = \frac{2\mathrm{i}}{\sin\pi(\mu-\bar\omega)}\{-\sin\pi 2\mu\sin\pi(\mu+\omega)s_{0t} + \sin\pi 2\omega_1\sin\pi(\mu+\bar\omega)r\}, \tag{4.26}$$

$$m_t = \frac{2\mathrm{i}(-1)^N\sin\pi(\mu+\omega)}{\sin\pi(\mu-\bar\omega)}\{\sin\pi 2\mu\,\mathrm{e}^{-\pi\mathrm{i}(\mu-\bar\omega)}s_{0t} - \sin\pi(\mu+\bar\omega)r\}, \tag{4.27}$$

$$m_1 = 2\mathrm{i}\sin\pi(\mu+\bar\omega)\,\mathrm{e}^{-\pi\mathrm{i}(N+2\mu)}r. \tag{4.28}$$

Proof. Comparison of the two differential equations for the σ-function, (1.3) and (3.11), imply that in general

$$\{v_1, v_2, v_3, v_4\} = \frac{1}{2}\begin{cases} \epsilon_1(\theta_t+\theta_\infty) \\ \epsilon_2(\theta_t-\theta_\infty) \\ \epsilon_3(\theta_0+\theta_1) \\ \epsilon_4(\theta_0-\theta_1), \end{cases} \tag{4.29}$$

with $\epsilon_j = \pm 1$, $j = 1, 2, 3, 4$ and $\epsilon_1\epsilon_2\epsilon_3\epsilon_4 = 1$. Using either set of parameters, (1.42) from the JUE correspondence or (1.38) from the CyUE correspondence, we find that the monodromy exponents can be given by one of three sets

$$\{\theta_0, \theta_t, \theta_1, \theta_\infty\} = \begin{cases} N+2\mu,\ N+2\omega_1,\ \mu+\omega,\ \mu+\bar\omega \\ N,\ N+2\mu+2\omega_1,\ \mu-\omega,\ \mu-\bar\omega \\ N+\mu+\omega,\ N+\mu+\bar\omega,\ -2\mu,\ 2\omega_1, \end{cases} \tag{4.30}$$

modulo permutations of the monodromy exponents and an even number of sign reversals. This is a manifestation of the non-uniqueness of the isomonodromic system for our problem. For definiteness we choose one example of the three cases, namely cases A, B and C given in (4.1), (4.2), (4.3). We note some simple identities which do not depend on the choice of the permutation or the sign

$$v_1^2+v_2^2+v_3^2+v_4^2 = \tfrac{1}{2}\left(\theta_0^2+\theta_t^2+\theta_1^2+\theta_\infty^2\right), \tag{4.31}$$

$$v_1v_2v_3v_4 = \tfrac{1}{16}\left(\theta_0^2-\theta_1^2\right)\left(\theta_t^2-\theta_\infty^2\right). \tag{4.32}$$

and in particular the following products which apply equally to cases (A), (B) and (C):

$$\left(\theta_0^2-\theta_1^2\right)\left(\theta_\infty^2-\theta_t^2\right) = (N-\mu+\omega)(N+\mu-\omega)(N+3\mu+\omega)(N+\mu+2\bar\omega+\omega), \tag{4.33}$$

$$\left(\theta_0^2-\theta_t^2\right)\left(\theta_\infty^2-\theta_1^2\right) = (N-\mu+\bar\omega)(N+\mu-\bar\omega)(N+3\mu+\bar\omega)(N+\mu+2\omega+\bar\omega), \tag{4.34}$$

$$\left(\theta_1^2-\theta_t^2\right)\left(\theta_\infty^2-\theta_0^2\right) = (2N+2\mu+2\omega_1)(2\mu-2\omega_1)(2\mu+2\omega_1)(\bar\omega-\omega). \tag{4.35}$$

If we make a comparison of the τ-functions themselves for the JUE correspondence, (3.10) and (1.41), we find, at the level of the τ-functions,

$$A_N(t;) = \tilde{C}t^{\frac{1}{8}(\theta_0^2+\theta_t^2-\theta_1^2-\theta_\infty^2)-\frac{1}{2}e_2[v^{\rm JUE}]-\mu N}(1-t)^{\frac{1}{8}(-\theta_0^2+\theta_t^2+\theta_1^2-\theta_\infty^2)-e_2'[v^{\rm JUE}]+\frac{1}{2}e_2[v^{\rm JUE}]}\tau(t;\theta). \tag{4.36}$$

Now applying the $t\to 0$ expansions for $\tau(t;\theta)$, namely (3.55), with those of $A_N(t;)$, (2.1), we first note that the exponent of the t-prefactors must be consistent and this implies

$$\sigma_{0t}^2 = \frac{1}{2}\left(\theta_0^2+\theta_t^2+\theta_1^2+\theta_\infty^2\right)+2e_2[v^{\rm JUE}] = \left(\sum_{i=1}^4 v_i^{\rm JUE}\right)^2 = (N-\mu+\bar\omega)^2. \tag{4.37}$$

This result applies to all the cases and for definiteness we make the choice of sign outlined in (4.4) (the other choice of sign is essentially equivalent). Turning to the leading analytic term of order t in both expansions, we find that its coefficient is given precisely by

$$\frac{2\mu(\mu+\omega)N}{N-\mu+\bar\omega}, \tag{4.38}$$

for all three cases upon employing our solution for σ_{0t}^2. Next we make a comparison of the non-analytic terms in (2.1) and (3.55) and it is here that we have to treat the cases separately. However, it is generally true that in the classical situation of the finite rank random matrix ensemble only one of the non-analytic terms is ever present, the other being switched off through the following mechanism. Taking case A first our task is to show how the coefficient of the non-analytic term $t^{1+\sigma_{0t}}$ vanishes and to match the remaining coefficient with that of the application. The identification of σ_{0t} immediately implies $\sigma_{0t}=\theta_0+\theta_t$. Therefore

$$[t^{1+\sigma_{0t}}] \propto \frac{1}{\Gamma\left(\frac{\theta_0+\theta_t-\sigma_{0t}}{2}\right)} = 0. \tag{4.39}$$

From a comparison of the coefficients of the remaining non-analytic term we find precise agreement and this enables us to determine the solution for the monodromy coefficient s_{0t} in (4.5). For case B, we note that $\sigma_{0t}=\theta_t-\theta_0$ and thus

$$[t^{1+\sigma_{0t}}] \propto \frac{1}{\Gamma\left(\frac{-\theta_0+\theta_t-\sigma_{0t}}{2}\right)} = 0. \tag{4.40}$$

Similarly we find agreement for the coefficient of the surviving non-analytic term and that enables us to fix the monodromy coefficient, as in (4.14). In case C we have $\sigma_{0t} = \theta_1 - \theta_\infty$ and thus

$$[t^{1+\sigma_{0t}}] \propto \frac{1}{\Gamma\left(\frac{-\theta_\infty+\theta_1-\sigma_{0t}}{2}\right)} = 0. \tag{4.41}$$

Again we find agreement for the coefficient of the $t^{1+\sigma_{0t}}$ term and conclude that the monodromy coefficient is given by (4.20). For each of the three cases, we observe that the Jimbo parameterization of the monodromy fails (see condition (3.22); however, a meaningful result emerges, so we conjecture that the theorem still holds with these relaxations.

Now we make a comparison of the expansions at $t = 1$, namely (2.2) and (3.60). Examination of the algebraic prefactor using the relation (4.36) leads us to conclude $\sigma_{t1}^2 = (2\mu + 2\omega_1)^2$ in all three cases and we choose the positive sign, as in (4.4). Employing this solution we compute the coefficient of the analytic term is, in all three cases,

$$\frac{\mu(\bar{\omega} - \omega)N}{2\mu + 2\omega_1}, \tag{4.42}$$

which is are entirely consistent with that in (2.2). To examine the non-analytic terms we take the three cases separately again. For case A, we see that the coefficient of the $(1-t)^{1-\sigma_{t1}}$ vanishes because $\sigma_{t1} = -\theta_0 - \theta_\infty$ and

$$[(1-t)^{1-\sigma_{t1}}] \propto \frac{1}{\Gamma\left(\frac{\theta_\infty+\theta_0+\sigma_{t1}}{2}\right)} = 0. \tag{4.43}$$

The coefficients of $(1-t)^{1+\sigma_{t1}}$ now agree precisely provided we have the solution (4.6) for the monodromy coefficient s_{t1}. For case B, we have the relation $\sigma_{t1} = \theta_1 - \theta_t$ and see that

$$[(1-t)^{1-\sigma_{t1}}] \propto \frac{1}{\Gamma\left(\frac{-\theta_1+\theta_t+\sigma_{t1}}{2}\right)} = 0. \tag{4.44}$$

Again the coefficients of $(1-t)^{1+\sigma_{t1}}$ agree and the solution (4.15) for s_{t1} follows. In case C we see that $\sigma_{t1} = \theta_\infty - \theta_0$ and this ensures

$$[(1-t)^{1-\sigma_{t1}}] \propto \frac{1}{\Gamma\left(\frac{-\theta_\infty+\theta_0+\sigma_{t1}}{2}\right)} = 0. \tag{4.45}$$

In this case we also find the coefficients of the remaining non-analytic terms are precisely consistent, leading us to deduce the solution (4.21) for s_{t1}.

It remains to make a comparison of the expansions at $t = \infty$, namely (2.3) and (3.61). Using (4.36) we see the algebraic pre-factor implies that $\sigma_{01}^2 = (N - \mu + \omega)^2$, and we choose the positive sign for the exponent. Using this value for σ_{01} we compute that the coefficient for the t^{-1} term in all three cases is

$$\frac{2\mu(\mu + \bar{\omega})N}{N - \mu + \omega}, \tag{4.46}$$

which is consistent with (2.3). To treat the non-analytic terms we take the cases separately. For case A, we note that $\sigma_{01} = \theta_t + \theta_\infty$ and this implies

$$[t^{-1-\sigma_{01}}] \propto \frac{1}{\Gamma\left(\frac{\theta_\infty+\theta_t-\sigma_{01}}{2}\right)} = 0. \tag{4.47}$$

The coefficient of the $t^{-1+\sigma_{01}}$ term is found to be in agreement with that of (2.3) if we take the solution (4.7) for s_{01}. For case B, the relation is $\sigma_{01} = \theta_t - \theta_\infty$ and this in turn implies

$$[t^{-1-\sigma_{01}}] \propto \frac{1}{\Gamma\left(\frac{-\theta_\infty+\theta_t-\sigma_{01}}{2}\right)} = 0. \tag{4.48}$$

Again exact agreement is found for the other coefficient provided that (4.16) holds. Lastly, in case C, we have the same relation as above and the absence of the $t^{-1-\sigma_{01}}$ term. Examination of the coefficients of $t^{-1+\sigma_{01}}$ then leads us to the solution (4.22).

Now we come to consideration of the connection relation (3.48) for $t = 0$ and its two equivalent forms for $t = 1, \infty$ with respect to our solutions for the monodromy data. We compute that the three connection relations of either sign decouple into a left-hand side and a right-hand side which vanish separately for all the cases A, B and C. The left-hand sides for the $t = 0$ connection relation vanish because $\theta_0 + \theta_t - \sigma_{0t} = 0$ and $\theta_\infty + \theta_1 + \sigma_{0t} = 2N$ for case A, $\theta_0 - \theta_t + \sigma_{0t} = 0$ and $\theta_0 + \theta_t + \sigma_{0t} = 2N$ for case B, and $\theta_\infty - \theta_1 + \sigma_{0t} = 0$ and $\theta_0 + \theta_t + \sigma_{0t} = 2N$ for case C. Similar reasoning applies to the connection relations at $t = 1$ and $t = \infty$. The right-hand sides of the relations vanish identically for both signs with the evaluations of $\sigma_{0t}, \sigma_{t1}, \sigma_{01}$, as given in (4.4).

To conclude we compute the monodromy matrices for the three cases and note that cases A, B and C yield the classical monodromy structure of lower triangular matrices, full matrices with one being a signed multiple of the identity, and upper triangular matrices, respectively. □

Remark 4.1. Our results are consistent with the findings of Mazzocco [20] which state that the classical non-algebraic solutions for P_{VI} have either reducible monodromy groups (cases A and C) or at least one monodromy matrix is equal to $\pm I$, that is the monodromy group is 1-smaller (case B). Both these cases cover the situation of a one-parameter (N) family of classical solutions.

Remark 4.2. We observe that the exponents $\sigma_{0t}, \sigma_{t1}, \sigma_{01}$ are not free boundary conditions for classical solutions but are fixed by certain combinations of the formal monodromy exponents. Related to this phenomenon is that all the connection relations decouple so that the coefficients of the monodromy coefficients s_{0t}, s_{t1}, s_{01} all vanish and thus cannot be determined from these relations. There is a geometrical picture of the classical solutions, which was discussed in relation to Painlevé II by Its and Kapaev [14]. The classical solutions of P_{VI} define singular points in the monodromy manifold which are characterized by $\mathfrak{M} = 0$ and

$$\frac{\partial}{\partial p_{0t}}\mathfrak{M} = p_{t1}p_{01} + 2p_{0t} - p_0p_t - p_1p_\infty = 0, \tag{4.49}$$

$$\frac{\partial}{\partial p_{t1}}\mathfrak{M} = p_{0t}p_{01} + 2p_{t1} - p_tp_1 - p_0p_\infty = 0, \tag{4.50}$$

$$\frac{\partial}{\partial p_{01}}\mathfrak{M} = p_{0t}p_{t1} + 2p_{01} - p_0p_1 - p_tp_\infty = 0. \tag{4.51}$$

We verify that these relations are satisfied for cases A, B and C.

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